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APPLICATION OF LINEAR PROGRAMING
to downward sloping demand problems in timber production

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Allocation models based on linear programing techniques are becoming widely used in scheduling timber harvests. The standard LP models must assume that stumpage prices are fixed. But this assumption is not valid if the timber producer's output is large enough to affect the market price. An optimum allocation can be approximated with linear programing by replacing the total revenue function with a piecewise linear function. The constant marginal revenues corresponding to each linear "piece" are then used as prices in the LP. The approximation to the true optimum can be improved by decreasing the range over which the marginal revenues are fixed. This simply requires shortening the linear segments used to approximate the total revenue function.

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The popularity of linear programing in analyzing timber production problems lies in its simplicity and ease of use–obtained, of course, at a price. Strong assumptions must be made about the technology of production and about the market in which the output is produced and sold. Specifically, the output produced is assumed to be proportional to the inputs used, and the price per unit of the output and the unit costs of the inputs are assumed to be independent of the quantity produced. When the producer's output is large enough to influence the price of the market, the assumption that the unit price is fixed or that the demand function is "horizontal" is clearly violated. A violated assumption can invalidate the results of any analysis based on this assumption.

In his classic text on linear programing, Dantzig[1] describes a method by which correct results can be obtained with linear programing when unit price *is* influenced by the quantity marketed. One condition must be met: the demand curve must be "downward sloping," that is, the price must fall as the quantity produced–and marketed–increases. The demand curve need not be "linear," but price–that is, average revenue (AR)–must fall with every increase in output (*fig. 1*).

This note describes the method and its application to the problem of analyzing the operation of a timber producer whose periodic output is large enough to influence the price at which the stumpage is sold. The Timber Resources Allocation Method (Timber RAM),[2] a linear programing planning model widely used by the Forest Service, is used in the illustration. Users of Timber RAM interested in specifying a falling demand schedule can request information and

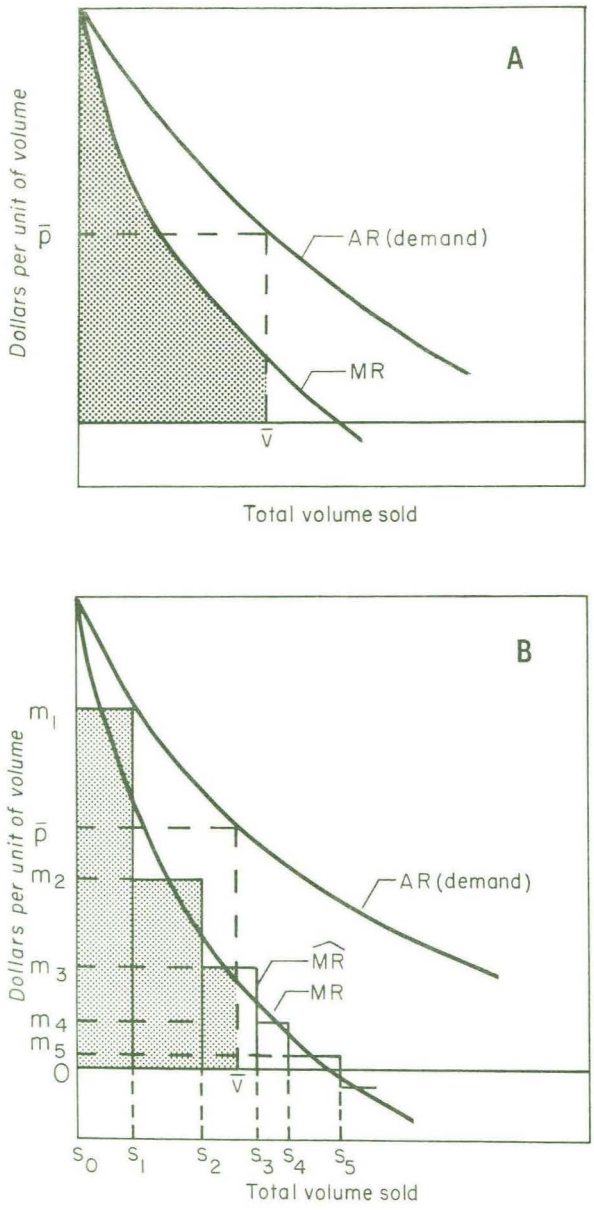

Figure 1—Two representations of total revenue: A, by the area under the marginal revenue function (MR) and B, by the area under a step function (MR) approximating marginal revenue.

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METHOD

A demand curve represents the combinations of price and quantity which will clear the market of a particular product. The total revenue of the seller for each combination is, of course, the product of the corresponding price and quantity.

The total revenue for a particular output $\bar{V}$ can also be calculated by measuring an area under the marginal revenue curve derived from the total revenue curve. This is the area bounded by the "marginal revenue" (MR) curve, the vertical axis, and the vertical line extended at $\bar{V}$ (*fig. 1a*). The marginal revenue curve represents the increase in revenue from selling an additional unit of output, at the corresponding output level. An approximation to this area can be easily calculated by drawing a histogram under the marginal revenue curve, measuring the height of each rectangle, multiplying it by the width of the base, and adding the areas of all the rectangles to the left of $\bar{V}$ (*fig. 1b*). The approximation procedure assumes that the marginal revenue represented by the height of the rectangle applies to the entire "subrange" of outputs spanned by the base of the rectangle.

Only the fraction of the rectangle in which $\bar{V}$ falls must be included in calculating the total revenue corresponding to $\bar{V}$. Note that the approximation to the actual total revenue can be improved by using narrower rectangles and that the width of the rectangles need not be equal.

The mathematical expression for this approximation to the total revenue at output $\bar{V}$ in *fig. 1b* is:

$$\hat{R}(\bar{V}) = m_1 s_1 + m_2(s_2 - s_1) + m_3(\bar{V} - s_2) \quad (1)$$

where: m_i is the marginal revenue corresponding to the volume sold over the subrange $s_i - s_{i-1}$;
$i = 1, 2, 3$
$s_0 = 0$

The actual total revenue at output $\bar{V}$ is the integral of the marginal revenue function from the origin to $\bar{V}$. By representing the volume harvested in the subrange $s_i - s_{i-1}$ with the dummy variable H_i, the approximation to total revenue $\hat{R}(\bar{V})$ can be expressed as a linear function in the variables H_i and can, therefore, be specified as the objective function in a linear program. To obtain the correct solution, however, the sum of the H_i must be equal to $\bar{V}$, and the linear program must drive each H_i to its maximum before including H_{i+1} in the solution (*fig. 1b*). These conditions are met by specifying two types of constraints in the linear program:

1. The level of H_i must be constrained to the width of the corresponding subrange. This is usually accomplished by specifying an "upper bound" on H_i rather than actually adding a row to the LP matrix.

2. The sum of the levels of the dummy variables (H_i) must be equal to V, the total volume harvested.

in mathematical notation:

$$H_i \leq s_i - s_{i-1} \qquad i = 1, 2, \ldots N \qquad (2)$$

$$\Sigma_i^N H_i = V \qquad (3)$$

ere N is the number of subranges specified.

The objective function maximized in the linear gram will be:

$$Z = \Sigma_i^N m_i H_i \qquad (4)$$

The optimality criterion always drives an activity its maximum before selecting another activity with wer payoff. By assumption, demand and marginal nue are falling. Hence $m_{i+1} < m_i$ and since m_i is payoff of activity H_i, this activity will be driven its upper limit by the simplex algorithm before $_{+1}$ is selected.

APPLICATION

Let us now illustrate how this technique can be plied to the analysis of stumpage production over eral time periods. For purposes of exposition, we ll apply the analytical technique to a representative blic forest in Oregon with a periodic harvest level ge enough to influence the stumpage price.[3] The lume harvested per acre and management costs are presentative of public forest management and flect the variations in species, age, site and logging actices typically encountered in the Pacific orthwest. These costs are assumed not to vary with e output level. A single demand curve is assumed r all the stumpage harvested in a given period. This emand curve is shifted outward every period to flect a rising secular trend in stumpage prices.

The problem is comparable to other forest anagement planning problems in terms of complexty and size and can be considered a valid test of the nalytical procedure. [4] The 620,000-acre land base is tratified into 55 timber classes. Inventory growth nd harvest volume for each timber class is etermined from one of 20 pre-defined yield chedules. All timber classes are managed under egimes including either a sequence of partial cuts and final harvest, or clearcutting only.

The objective is to maximize the total discounted net revenue obtainable over a span of decades. The LP objective function is specified as the difference between total revenues reflecting a falling demand function for the total stumpage sold in each period and the Timber RAM cost objective function. Volume harvested may be constrained over additional decades to insure a specified minimum level of harvest in the future. A falling demand schedule is specified for the first decade. The demand schedule for this example is derived on the basis of a demand analysis on the adjoining National Forest land by Bell[5] and Adams.[6] It is not intended to be an empirically accurate portrayal of the price/quantity relationship for stumpage. Rather, the intent is only to illustrate the application of the method. The base demand schedule is:

$$P = 1{,}500 - .001479(Q) \qquad (5)$$

where P is the average price in constant dollars per thousand cubic feet (MCF) and Q is the total volume sold in the first decade.

The corresponding marginal revenue function having the same intercept and twice the slope is:

$$MR = 1{,}500 - .002558(Q) \qquad (6)$$

Demand schedules for subsequent decades are specified by shifting out the base demand schedule by 15 percent per decade at the point of unitary elasticity. For each decade, the falling demand schedule is interpreted as follows: the schedule indicates the average price which will be paid for stumpage during each year of the decade, given the total volume of stumpage marketed during the decade. The demand schedule is only indicative—rather than predictive—of the behavior of a stumpage market. It nevertheless provides useful information for long-range planning and policy analysis, particularly since it abstracts from seasonal and annual disturbances.

On the basis of historical trends, the maximum anticipated harvest level is set in terms of total output per decade. The range from 0 to this maximum harvest level is partitioned into unequal subranges; m_i^t, the marginal revenue for subrange i in decade t, is defined as the value of the marginal revenue function at the midpoint of the i^{th} subrange.

In forestry, the analysis of production typically spans many decades. The estimates of the corresponding discounted revenues and costs, however, need only be made up to the time when the interest rate has reduced revenues and costs to a small percentage of their undiscounted value. For instance, $100 discounted at 8 percent annually is reduced in value to $4.50 after 40 years, to 98 cents after 60 years, and to 21 cents after 80 years. Individuals will disagree on the exact time when the financial analysis should be conducted. It is clear that the analysis from a financial standpoint need not be extended beyond 60 years from the present. However, nonfinancial criteria may justify a longer period of analysis. If a

Table 1–*Parameter levels influence both the optimal solution and the computing time*

Parameters	Computer-run					
	1	2	3	4	5	6[1]
Subranges, no.	6	6	12	6	12	–
Decades of maximization, no.	12	12	12	30	30	–
Harvest floor, MMCF	none	350	none	350	350	–
Discounted net revenue of harvest schedule, dollars	476,326.6	470,888.5	478,295.9	470,347.5	473,745.1	460,145.5
Computing time, system seconds	256	294	257	338	328	198

[1] Current policy run: nondeclining yield in the conversion period. Price/MCF, inflated at 15 percent per decade, is not a function of output.

Table 2–*Step functions approximating the initial marginal revenue curve are defined in terms of fixed prices for each sub-range*

Subranges	Runs 1, 2, and 4		Runs 3 and 5	
	M_i	Output range	M_i	Output range
	Dollars	*MMCF*	*Dollars*	*MMCF*
1	1200.00	0–200	1352.10	0– 100
2	760.50	201–300	1056.30	101– 200
3	464.70	301–400	834.45	201– 250
4	168.90	401–500	686.55	251– 300
5	–126.90	501–600	538.65	301– 350
6	–570.60	601–800	390.75	351– 400
7	–	–	270.00	401– 433
8	–	–	168.90	434– 466
9	–	–	71.30	467– 500
10	–	–	–52.90	501– 550
11	–	–	–200.80	551– 600
12	–	–	–570.60	601–1000

period of analysis (span of decades in which discounted net revenue is maximized) produces unacceptable harvest levels in subsequent decades, the period of analysis may be lengthened beyond the financial limit.

The illustration consists of five computer runs representing variations in three basic parameters: the number of subranges, the number of decades to be maximized, and the inclusion of a floor on the level of harvests (*table 1*). For runs 2, 4 and 5, the periodic harvest floor was set at 350 MMCF per decade. For runs 3 and 5, the marginal revenue curve was approximated more accurately by specifying smaller subranges over the region of likely harvest (*table 2*).

ANALYSIS OF RUNS

In the absence of inter-temporal harvest flow constraints, the optimal cutting pattern is characterized by steeply declining cuts over the first several decades (*figs. 2a and 2c*). Increases in harvest level from periods 1 to 2 result from the interaction between subrange limits, growth, discount rate, and price increases from shifts in the MR curve. This flow pattern is a distinct alternative to the nondeclining even-flow harvest patterns currently implemented on most public forest lands in the West under present policy of biological optimization subject to strong harvest constraints (*fig. 2f*). Nondeclining yield harvest constraints often force harvest levels in the conversion period lower than the financial optimum. For most public forest land in the West, the unconstrained maximization of discounted total net revenue, with a downward sloping demand curve, would generally produce harvest schedules with a higher present value than schedules produced under current, biologically oriented policy (*table 1*). In timber markets with highly elastic demand for stumpage, however, present value might not be higher and optimal harvest levels might be lower.

Smaller subranges will reduce the error in the approximation of gross revenues and consequently of net revenue. Additional subranges increase the processing costs modestly (*table 1*). With more subranges the optimal harvest schedule changes slightly, but the total discounted net revenue is virtually unchanged (i.e., runs 1 and 3, *figs. 2a and 2c*). This result suggests that even a crude stepwise approximation to the marginal revenue schedule may provide an adequate analysis of harvest alternatives.

The fluctuation in periodic harvests brought on by the unconstrained maximization of discounted net revenue may not be administratively or politically feasible. Constraining the solution with a floor on the harvest levels reduces the variability in periodic harvest levels with some sacrifice in present value of the total schedule (runs 3 and 5, *figs. 2c and 2e*). Computing costs are more sensitive to the addition of

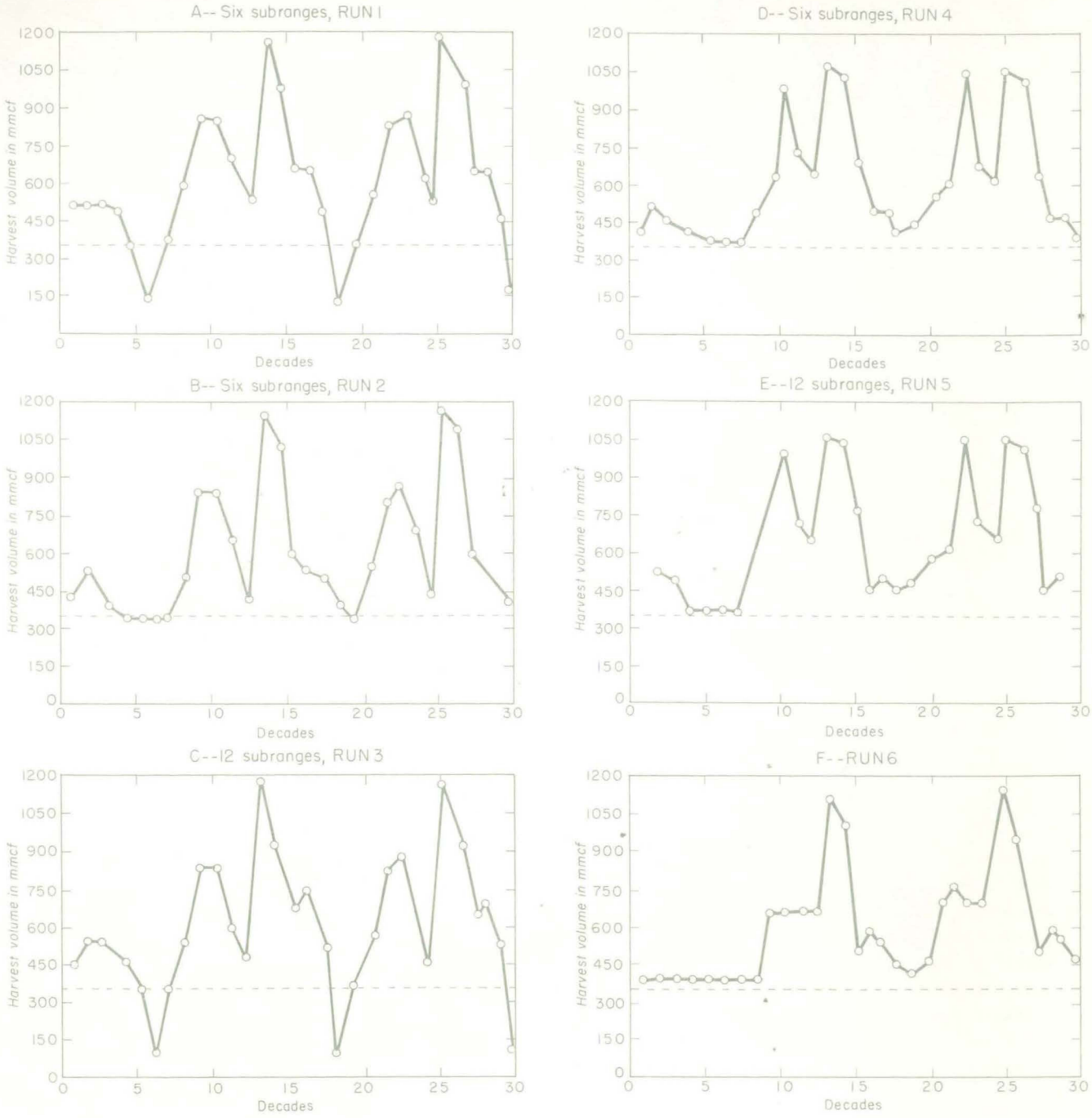

Figure 2—A harvest floor influences the optimal solution throughout the planning period. Because of discounting, the number of subranges and the period of optimization primarily influence the solution in the initial decades. *A,* no harvest floor, maximize over 12 decades. *B,* harvest floor at 350MMCF, maximize over 12 decades. *C,* no harvest floor, maximize over 12 decades. *D,* harvest floor at 350MMCF, maximize over 30 decades. *E,* harvest floor at 350MMCF, maximize over 30 decades. *F,* present policy: nondeclining even-flow over the conversion period.

flow constraints than to additional subranges (*table 1*). Even higher minimum periodic harvest constraints can be imposed if a greater reduction in the fluctuation of periodic harvests is desired.

The harvest schedule is also influenced by the number of decades over which discounted net revenue is maximized (runs 2 and 4, *figs. 2b and 2d*). However, in this illustration the difference is barely discernible because both 12 decades and 30 decades are relatively long periods of analysis. Computing costs increase with a longer period of analysis (*table 1*). Preparation time is not increased, however, because marginal revenues, m_i^t can be generated by a utility program. It is noteworthy that for this data base and demand curve, the discounted net revenue of the optimal harvest schedules based on downward sloping demand do not differ significantly from the discounted net revenue optimal harvest schedule generated under present policy (run 6, *fig. 2f*).

CONCLUSIONS

The application of linear programing allocation models, such as Timber RAM, to problems of downward sloping demand schedules is theoretically valid and operationally feasible. By approximating the marginal revenue curve corresponding to any downward sloping demand curve with a step function, linear programing models can approximate the total revenue at any output level. The accuracy of the approximation is readily improved by increasing the number of steps used to represent the marginal revenue curve. Computing costs increase as the number of subranges is increased. However, this limited test suggests that increases in computing costs are not prohibitively high. Harvest scheduling models using linear programing can provide a sound and economical method for analyzing silvicultural alternatives when output does affect stumpage price.

NOTES

[1] Dantzig, George B. 1963. *Linear programming and extensions.* p. 482-485. Princeton University Press. Princeton, N.J.

[2] Navon, Daniel I. 1971. *Timber-RAM, a long-range planning method for commercial timber lands under multiple-use management.* USDA Forest Serv. Res. Paper PSW-70, 22p, illus. Pacific Southwest Forest and Range Exp. Stn., Berkeley, Calif.

[3] Data compiled by Pacific Northwest Region, Forest Service, U.S. Department of Agriculture, Portland, Oreg.

[4] We are indebted to Larry J. Strauss for writing the utility program used to extend the standard Timber RAM LP tableau, for his work in carrying out the computer runs, and for many helpful suggestions.

[5] Bell, Enoch. 1975. (Unpublished data on file at Pacific Northwest Forest and Range Exp. Stn., Portland, Oreg.)

[6] Adams, Darius M. 1975. *Response of softwood stumpage and end product prices to changes in National Forest harvest schedules.* (Unpublished report on file at Pacific Northwest Forest and Range Experiment Station, Portland, Oreg.)

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